P. J. ŠAFÁRIK UNIVERSITY
FACULTY OF SCIENCE
INSTITUTE OF MATHEMATICS
Jesenná 5, 040 01 Košice, Slovakia

K. Cechlárová and E. Pillárová

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Katarína Cechlárová and Eva Pillárová
Institute of Mathematics, Faculty of Science, P.J. Šafárik University,
Jesenná 5, 040 01 Košice, Slovakia
e-mail:katarina.cechlarova@upjs.sk, eva.pillarova@student.upjs.sk

Abstract

Let the cake be represented by the unit interval of reals, with players having private valuations expressed by nonatomic probability measures. The aim is to find a cake division which assigns to each player one contiguous piece (a simple division) in such a way that the value each player receives (by her own measure) is the same for all players. It is known that such divisions always exist, however, we show that there is no finite algorithm to find them already for three players. Therefore we propose an algorithm that for any given $\varepsilon > 0$ finds, in a finite number of steps, a simple division such that the values assigned to players differ by at most $\varepsilon > 0$.

Keywords. Cake cutting, equitable division, algorithm, approximation.

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JEL classification. C72, D63

1 Introduction

In this paper we deal with the problem of 'fairly' dividing a certain resource, called the cake, between n people (players). The cake is represented by the interval $[0, 1]$ of reals. Players have different opinions about the values of different parts of the cake. We shall suppose that these valuations are private information of players.

Although people have been trying to divide things 'fairly' for a very long time, a rigorous mathematical theory of fair division was established only after the second world war [12]. We shall concentrate on equitable divisions, i.e. such that the values of pieces assigned to all players are equal (according to their valuations). In the literature, other concepts of fairness are considered, too. In

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a proportional division (sometimes called simple fair [11]) each player receives at least $1/n$ part of the cake according to her valuation, in an envy-free division no player thinks that she would be better off with somebody else's piece and an exact division assigns pieces such that everybody thinks that everybody's piece has value exactly $1/n$. It is known that in general, these properties are not equivalent, but exactness implies all the other properties (see e.g. [3] and [11], where also some other notions are defined and the relations between them explored).

Equitability is not so popular as proportionality or envy-freeness. Existence of exact divisions was proved by Dubins and Spanier in [8] (however, the obtained pieces could be any members of a σ algebra on $[0, 1]$) and by Alon [1], who showed that in the worst case for n players as many as $n(n-1)$ cut points may be necessary. Such divisions may be very impractical in the real life. Imagine researchers who share a very expensive apparatus needed for their experiments. If they had to come into the lab and leave it several times a day, they would perhaps rather give up such a kind of fairness. Therefore we are interested in cake divisions where each player receives a contiguous piece. Such cake divisions will be called *simple* in this paper. They are specified by their cutpoints and the order of players. Simple equitable divisions were studied by Mawet, Pereira and Petit [9] for piecewise constant utility functions and by Aumann and Dombb [2], who used the compactness of the set of all simple divisions. Cechlárová, Doboš and Pillárová [7] proved the existence of simple equitable divisions for any number of players in any order.

In general, it is easier to prove the existence of a division fulfilling a certain property than to find such a division, see a nice review in [11], Chapter 7. In recent years, several papers studied what can be achieved by a finite algorithm. A finite cake cutting algorithm, as specified by [11], [16] or [14], uses a finite number of requests of two types issued to players:

- 'For a given value $\alpha \in [0, 1]$, determine the smallest point x such that your value of the interval $[0, x]$ is equal to α!' (cutting query)
- 'What is your value of the given cake piece?' (evaluation query)

With a little thought it is clear that the beginning of the interval in the first type of request need not be fixed at 0. We shall also use the third kind of request:

- 'For a given value $\alpha \in [0, 1]$, determine the biggest point x such that your value of the interval $[x, 1]$ is equal to α!' (modified cutting query)

However, what is important, a finite algorithm does not require the knowledge of complete value functions of players. Also the famous moving-knife algorithms [13], [4] cannot be considered finite. There are even results proving that no finite algorithm can exist for finding divisions of a certain type. Robertson and Webb [10] proved that there exists no finite algorithm that produces an exact division for two players. Stromquist [14] showed that neither an envy-free simple division among three players can be obtained by a finite algorithm.

Hence algorithms that produce a 'nearly fair' division are called for. Robertson and Webb [10], [11] provided a finite algorithm that, given a small $\varepsilon > 0$ and a set of real numbers $\alpha_1, \cdots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$, constructs a division such that for each player i, the value of her piece differs from α_i by at most ε. The main idea of the algorithm is the following. Player 1 cuts the cake into pieces which she considers to be smaller than k each, where k is a small number determined by ε and n; in the case of two players k may be set to $\varepsilon/2$. Then player 2 can reduce any of the pieces (if necessary) so that each new piece will be smaller than k according to her, etc. An ε-exact division is then produced by a suitable assignment of the obtained pieces to players. A disadvantage of this algorithm is that many small pieces arise and those assigned to one player can be scattered irregularly over the whole cake.

Another ε-exact division for two players can be obtained using the approach described by Simmons and Su in [15]. They considered the so-called consensus-halving, i.e. a division of an object into two portions so that each of n people believes the portions are equal. (If $n = 2$, an exact division is obtained.) Simmons and Su showed, using methods from combinatorial topology, namely theorems of Borsuk-Ulam and Tucker, that such a division exists, at most n cuts are needed and this number of cuts is the best possible. Moreover, they showed how a constructive proof of Tuckers lemma yields a finite algorithm for locating an ε-approximate solution that uses the minimal number of cuts.

In this paper we show that no finite algorithm can find a simple equitable division for three players (and give a hint to a proof for an arbitrary number of players). We obtain this results by a modification of Stromquist's work [14], by constructing *stiff measure systems.* A finite algorithm for an ε-equitable simple division for two players was proposed by Cechlárová and Pillárová [6]. Here we construct a finite algorithm that finds a near equitable simple division for any number of players.

2 Definitions and basic properties of divisions

We will consider the set of *players* $N = \{1, 2, ..., n\}$. The *cake* is represented by the interval $[0, 1]$. In this work, the only *allowable portions – pieces* are intervals $[p, q]$, $0 \leq p \leq q \leq 1$. A cutpoint of two neighbouring pieces cannot belong to both of them, but since in our model the value of a piece is not influenced by a single point, we shall represent all pieces as closed intervals.

We shall suppose that each player i is endowed with a nonatomic probability measure U_i on the cake. Such a measure can be represented by the distribution function $F_i(x) = U_i(0, x)$, so that the measure of each interval $[p, q]$ is equal to $F_i(q) - F_i(p)$. The properties of the measure imply that the function F_i is nonnegative, nondecreasing and continuous on [0,1] and $F_i(0) = 0, F_i(1) = 1$. If the distribution function F_i has a density f_i, then

$$U_i(p, q) = \int_p^q f_i(t)dt.$$

(Note that the terminology is perhaps not very intuitive in the context of cake cutting, however, the properties of distribution functions and their densities are so well-known that we shall stick at it.)

A cake division D is a partition of the cake into n disjoint pieces; the piece assigned to player i in a division D will be denoted by D_i. The various fairness criteria are formulated in the following definition (see also [3, 11] for other notions and relations between them).

Definition 1 *A cake division $D = (D_1, D_2, \dots, D_n)$ is said to be*
a) proportional, if $U_i(D_i) \geq 1/n$ for each $i \in N$
b) exact, if $U_i(D_j) = 1/n$ for each $i, j \in N$
c) envy-free, if $U_i(D_i) \geq U_i(D_j)$ for each $i, j \in N$
d) equitable, if $U_i(D_i) = U_j(D_j)$ for each $i, j \in N$.

Simple cake divisions are specified by their cutpoints and the order of players.

Definition 2 *A simple cake division is a pair $D = (d, \varphi)$, where d is an $(n-1)$-tuple $(x_1, x_2, \dots, x_{n-1})$ of cutpoints with $0 \leq x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq 1$, and $\varphi : N \to N$ is a permutation of N.*

For technical reasons, we set $x_0 = 0$ and $x_n = 1$. Permutation φ in the division D means that player i is assigned the interval $[x_{j-1}, x_j]$ if $\varphi(j) = i$. In what follows, we abbreviate the phrase *equitable simple division* by ESD. The common value that each player receives in an ESD D will be denoted by $E(D)$.

Cechlárová, Doboš and Pillárová [7] proved the following assertions:

Theorem 1 *For any number of players n there exists an ESD for each players' order. If the probability density function of each player is everywhere strictly positive then in the given players' order the ESD is unique.*

The following assertion is mainly technical, but it has important consequences.

Lemma 1 *Let D' be any simple cake division for n players with players' order π such that player i receives a piece with value $U_i(D'_i)$. Then any ESD D with players' order π fulfills*

$$min\{U_j(D'_j), j = 1, 2, \dots, n\} \leq E(D)) \leq max\{U_j(D'_j), j = 1, 2, \dots, n\}.$$

Proof. The first inequality was proved in [7]; we give here an analogic proof for the second one.

Let us suppose, without loss of generality, that π is the identity permutation. Denote the cutpoints of the division D' by $(d_1, d_2, \dots, d_{n-1})$ and the cutpoints of the equitable cake division D in the players' order π by $(e_1, e_2, \dots, e_{n-1})$. Since D is equitable, it suffices to show that $U_i(D_i) \leq U_i(D'_i)$ for some i.

We distinguish three cases:

(a) $e_1 \leq d_1$. Then $[0, e_1] \subseteq [0, d_1]$ and so $U_1(D_1) \leq U_1(D'_1)$.

(b) If there exists k such that $e_j > d_j$ for each $j = 1, 2, ..., k-1$ and $e_k \leq d_k$ then $[e_{k-1}, e_k] \subset [d_{k-1}, d_k]$. Hence $U_k(D_k) \leq U_k(D'_k)$.

(c) $e_j > d_j$ for each $j = 1, 2, \dots, n-1$. Then $[e_{n-1}, 1] \subset [d_{n-1}, 1]$ and therefore $U_n(D_n) \leq U_n(D'_n)$.

■

We shall formulate two corollaries of Lemma 1. The first one claims that taking an ESD in a players' order in which a proportional simple division exists ensures that the ESD is also proportional. The second one says that even in the case when there are several different ESDs in one players' order, the utility that they assign to players is the same.

Corollary 1 *If D' is a proportional simple division and D an ESD with the same players' order then D is also proportional.*

Corollary 2 *If D and D' are two ESDs with the same players' order then $E(D) = E(D')$.*

3 A finite algoritm does not exist

Although equitable simple divisions are sure to exist, they are not so simple to find. We give an independent proof of the nonexistence of a finite algorithm capable of finding a proportional ESD. Our proof resembles the one given by Stromquist [14] for the envy-free simple division for three players.

A *stiff measure system* (SMS for short) for three players L, M, R (for Left, Middle, Right) is a triple of probability densities u_L, u_M, u_R, defined by four parameters s, t, x, y, where $0 < s < 1/6$, $2s + t = 1$ and $0 < x < y < 1$. The probability densities u_L, u_M, u_R are everywhere positive and such that the values of intervals correspond to Table 1.
The basic properties of a SMS are described in the following Lemma.

Lemma 2 *If the three players L, M, R have a SMS with parameters s, t, x, y then the only proportional ESD has the players' order (L, M, R) and cuts in points x and y.*

	$[0, x]$	$[x, y]$	$[y, 1]$
U_L	t	s	s
U_M	s	t	s
U_R	s	s	t

Table 1: Values in a stiff measure system

Proof. First we show that the only possible players' order in a proportional simple division is (L, M, R).

Should player R be first, then she must get an interval $[0, z]$ such that $z \geq y$. However, the value of the remaining piece is smaller than $1/3$ both for M as well as for L. If M were first, then she must get an interval $[0, z]$ such that $z > x$. Then player L could not get a piece with value at least $1/3$.

Hence we have that in any proportional simple division the first player must be player L. If player M were last, then her piece must be of the form $[z, 1]$ where $z < y$. Then the piece that remains for player R would have a value smaller than $1/3$.

Therefore the only possible players' order is (L, M, R). Clearly, the cutpoints $x_1 = x$ and $x_2 = y$ give everybody a piece of value t. Now Theorem 1 is sufficient for the uniqueness of the proportional ESD in this order. ■

The following lemma is an analogy of Lemma 3 of [14]. Its interpretation is following: even if one player knows that the three probability densities form a SMS and she knows her probability density in full and the densities of other players outside the neighbourhoods of suspected cutpoints, she alone is not able to determine the parameters of the SMS. We formulate the assertion for player L, but analogic results hold for players M and R.

Lemma 3 *Let u_L, u_M, u_R form a SMS with parameters s, t, x, y. Let $\eta > 0$ be arbitrary, $\delta > 0$ be such that $2/3 + \delta < t < 1 - \delta$, and let $\hat{t} \neq t$ be sufficiently close to t. Then there exists a SMS $\hat{u}_L, \hat{u}_M, \hat{u}_R$ with parameters $\hat{s}, \hat{t}, \hat{x}, \hat{y}$ for some $\hat{s}, \hat{x}, \hat{y}$ such that*

(*i*) $\hat{u}_L(z) = u_L(z)$ *for all* $z \in [0, 1]$,

(*ii*) $\hat{u}_j(z) = u_j(z)$ *for all* $j \neq L$ *and all* z *that are outside the* η*-neighbourhoods of* x *and* y,

(*iii*) $\hat{x} \neq x$, $\hat{y} \neq y$, *but both are within* η*-neighbourhoods of* x *and* y, *respectively;*

(*iv*) $2/3 + \delta < \hat{t} < 1 - \delta$

Proof. Our proof will do the following: The choice of $\hat{t}$ (we will see later, how close to t this should be) will uniquelly determine the value of $\hat{s}$ and the new cutpoints $\hat{x}$ and $\hat{y}$, so as the probability density u_L (since it has not changed) will fulfill the requirements of the first row of Table 1 for t, s, x, y replaced by $\hat{t}, \hat{s}, \hat{x}$ and $\hat{y}$.

Suppose that $\hat{t} < t$ (the case $\hat{t} > t$ can be treated similarly). This implies $\hat{s} > s$ and $\hat{x} < x$, $\hat{y} < y$. Given $\eta > 0$, we recall that $U_L(0, x)$ is a nondecreasing continuous function of x, so $\hat{t}$ can be chosen in such a way that it fulfills (iv) and $\hat{x}, \hat{y}$ fulfill (iii).

Now it is sufficient to show that there exist probability densities $\hat{u}_M$ and $\hat{u}_R$, identical with the probability densities u_M and u_R outside the η-neighbourhoods of x and y such that together with u_L they form a SMS with parameters $\hat{s}, \hat{t}, \hat{x}$ and $\hat{y}$. Namely, for player M we have that the following relations have to be fulfilled:

$$\hat{U}_M(0, \hat{x}) = \hat{s},\ \hat{U}_M(\hat{x}, \hat{y}) = \hat{t},\ \hat{U}_M(\hat{y}, 1) = \hat{s}. \tag{1}$$

Let us show that $\hat{u}_M$ can be made constant in the intervals

$$(x - \eta, \hat{x}),\ (\hat{x}, x + \eta), (y - \eta, \hat{y}),\ (\hat{y}, y + \eta), \tag{2}$$

so that the measure $\hat{U}_M$ of player M fulfills row 2 of Table 1 (with the new parameters). Let us denote the values of these constants by m_1, m_2, m_3 (notice that m_2 is the same for both middle intervals). Taking into account that $\hat{u}_M = u_M$ outside the η-neighbourhoods of x and y, we have the following constraints:

$$\left.\begin{array}{rcl} U_M(0, x-\eta) + m_1(\hat{x} - (x - \eta)) &=& \hat{s} \\ U_M(x+\eta, y-\eta) + m_2((x + \eta - \hat{x}) + (\hat{y} - (y - \eta))) &=& \hat{t} \\ U_M(y+\eta, 1) + m_3(y + \eta - \hat{y}) &=& \hat{s}. \end{array}\right\} \tag{3}$$

To ensure that m_1, m_2, m_3 can be positive, it is necessary and sufficient to have

$$U_M(0, x-\eta) < \hat{s},\ U_M(x+\eta, y-\eta) < \hat{t},\ U_M(y+\eta, 1) < \hat{s}. \tag{4}$$

The first and the third conditions in (4) hold trivially because $U_M(0, x - \eta) < s < \hat{s}$ and $U_M(y + \eta, 1) < s < \hat{s}$. For the second one let us realize that $U_M(x + \eta, y - \eta) < t$, so it suffices to choose $\hat{t}$ in the interval $[U_M(x + \eta, y - \eta), t]$ (this is the second condition showing how close $\hat{t}$ should be to t).

Analogically, for player R we get the conditions

$$\hat{U}_R(0, \hat{x}) = \hat{s},\ \hat{U}_R(\hat{x}, \hat{y}) = \hat{s},\ \hat{U}_R(\hat{y}, 1) = \hat{t}. \tag{5}$$

We shall make $\hat{u}_R$ constant in the intervals (2) so that the measure of player R fulfills row 3 of Table 1 (with the new parameters). Let us denote the values of these constants by r_1, r_2, r_3 (again, r_2 is the same for both middle intervals). We have the following constraints:

$$\left.\begin{array}{rcl} U_R(0, x-\eta) + r_1(\hat{x} - (x - \eta)) &=& \hat{s} \\ U_R(x+\eta, y-\eta) + r_2((x + \eta - \hat{x}) + (\hat{y} - (y - \eta))) &=& \hat{s} \\ U_R(y+\eta, 1) + r_3(y + \eta - \hat{y}) &=& \hat{t}. \end{array}\right\} \tag{6}$$

r_1 and r_3 will be positive since

$$U_R(0, x-\eta) < s < \hat{s},\ U_R(x+\eta, y-\eta) < s < \hat{s}. \tag{7}$$

To ensure that $U_R(y+\eta, 1) < \hat{t}$ it suffices to choose $\hat{t}$ so that it lies in the interval $[U_R(y+\eta, 1), t]$ (the last condition for $\hat{t}$ to be close to t).

Hence the new probability densities will be everywhere positive and they will fulfil the requirements of the lemma. ■

Theorem 2 *There is no finite algorithm for finding a proportional ESD for three players. This assertion remains true even for three players whose probability densities form a SMS.*

Proof. Suppose, on the contrary, that there is such an algorithm. Suppose that the probability densities u_L, u_M, u_R of the three players L, M, R form a SMS. We will show that after any finite number of steps, the algorithm is not able to determine precisely the parameters of the SMS, in particular, to determine the cutpoints of the unique ESD.

At the beginning, suppose that the parameters of the SMS u_L, u_M, u_R are s, t, x, y. Suppose that the algorithm has already performed K steps, where K is any positive integer. If no mark has been made at x or y, the algorithm proceeds to step $K+1$. On the other hand, if the algorithm made a mark at say x, Theorem 3 implies that all the marks made so far could have as well been obtained for another SMS $\hat{u}_L, \hat{u}_M, \hat{u}_R$ with parameters $\hat{s}, \hat{t}, \hat{x}, \hat{y}$: it suffices to take η such that no mark is within the η-neighbourhood of x and y. So the algorithm has not found the correct cutpoints and it has to continue by another step. ■

The technique used in this section can easily be extended to any number of players $n \geq 3$. Let s be any positive number such that $s < 1/n(n-1)$. We define $t = 1-(n-1)s$. Obviously, $t > (n-1)/n > 1/n$. There will be points $x_1, x_2, \ldots, x_{n-1}$ and the players' measure will fulfil

$$U_i(x_{j-1}, x_j) = \begin{cases} s & \text{if } j \neq i \\ t & \text{if } j = i \end{cases}$$

It can be shown that the only players' order ensuring proportionality is $(1, 2, \ldots, n)$ – it suffices to realize that for any $i < n$, if a player $j > i$ were in position i, then her piece should end in a point $y > x_j$, leaving not enough for the rest of players. In this order there is again a unique ESD with cutpoints $x_1, x_2, \ldots, x_{n-1}$ that cannot be found in a finite number of steps.

4 The near equitable algorithm

The algorithm that we propose in this section finds a simple cake division such that the difference between the values of pieces assigned to players is not higher than a predetermined value ε. We call this property of a cake division ε-equitablity:

Definition 3 *Let $D = (d, \varphi)$ be a simple cake division and $\varepsilon > 0$ a real number. D is called ε-equitable if*

$$|U_{\varphi(j)}(x_{j-1}, x_j) - U_{\varphi(k)}(x_{k-1}, x_k)| \leq \varepsilon \text{ for each } j, k \in N.$$

The algorithm consists of three phases:
Phase 1. `Ordering`
Phase 2. `Addition`
Phase 3. `Termination`

4.1 `Ordering`

The purpose of Phase 1 is to find a players' order, in which a proportional simple division exists. There are several finite algorithms to find a proportional simple division for n players, we describe here a simplified version of the procedure proposed in Chapter 4 of [11].

First, each player i is asked to mark a point y_i such that $U_i(0, y_i) = 1/n$. The player, let us call her i_1, whose mark is most to the left is chosen. (In case when the left-most marks correspond to several players, any one of them can be taken.) The mark made by player i_1 is denoted by x_1. Player i_1 is assigned the piece $[0, x_1]$ and she drops out of the game. x_1 is taken as the new beginning of the cake and the marks made so far are erased. Each player i of the remaining $n-1$ players is asked to make a new mark at a point y_i such that $U_i(x_1, y_i) = 1/(n-1)$. Again, the player, say i_2, with the left-most mark, denoted now by x_2 is taken etc. until just one player remains, who is assigned the rest of the cake. The obtained players' order is $(i_1, i_2, \ldots, i_n)$ and it is easy to see that in the simple division with the cuts in points $x_1, x_2, \ldots, x_{n-1}$ each player receives a piece that she considers to have a value at least $1/n$.

Thanks to Corollaries 1 and 2 we know that any ESD taken with the players' order determined in Phase 1 will give each player a piece with the same value $E \geq 1/n$.

4.2 `Addition`

In Phase 2 the algorithm actually tries to approximate the value E by constructing its *binary expansion.* Recall that a binary expansion of a real number $E \in [0, 1]$ is its expression in the form

$$E = \sum_{k=0}^{\infty} c_k.1/2^k \tag{8}$$

where $c_k \in \{0,1\}$ for each k.[1] The j^{th} *partial binary expansion* is

$$\lfloor E \rfloor_j = \sum_{k=0}^{j} c_k . 1/2^k \tag{9}$$

Here we describe in detail just the case with 3 players and later we generalize these ideas to an arbitrary number of players. Rename the players L, M, R according to the order obtained in Phase 1.

Phase 2 works in iterations. The piece of player L begins in point 0, the pieces of players M and R immediately follow. As the algorithm proceeds, the values of pieces for players L and M are kept mutually equal, and the right player R is ensured the value that is not smaller.

In iteration j we want to decide whether $c_j = 1$ or $c_j = 0$ on assumption that the $(j-1)^{st}$ partial binary expansion $\lfloor E \rfloor_{j-1}$ has so far been correctly determined. In other words, all three players already have for sure a piece with value $\lfloor E \rfloor_{j-1}$ and we want to know whether their pieces can be enlarged by $1/2^j$. We do it by issuing three requests:

1. To player L: Tell the smallest x_1 such that $U_L(0, x_1) = \lfloor E \rfloor_{j-1} + 1/2^j$.
2. To player M: Tell the smallest x_2 such that $U_M(x_1, x_2) = \lfloor E \rfloor_{j-1} + 1/2^j$.
3. To player R: Is $U_R(x_2, 1) \geq \lfloor E \rfloor_{j-1} + 1/2^j$?

(Notice that it may happen that $U_M(x_1, 1) < \lfloor E \rfloor_{j-1} + 1/2^j$. In that case the player is instructed to output $x_2 = 1$.) We say that iteration j is *successful* if the following condition holds:

$$U_R(x_2, 1) \geq \lfloor E \rfloor_{j-1} + 1/2^j \tag{10}$$

otherwise the iteration is said to be *unsuccessful.* Connection between the success in an iteration and the corresponding digit in the binary expansion of E is described in the following Lemma.

Lemma 4 *For each j, the value of digit c_j in the binary expansion of the equitable value E for the players' order (L, M, R) determined by the rule*

$$c_j = \begin{cases} 1 & \textit{if iteration } j \textit{ is successful} \\ 0 & \textit{otherwise} \end{cases} \tag{11}$$

is correct.

[1] Notice that some reals have two different binary expansions. Namely the two expansions c_k and c'_k are such that there exists j with $c_j = 1$ and $c_k = 0$ for each $k > j$ and $c'_j = 0$ and $c'_k = 1$ for all $k > j$ (compare e.g. [5]). Due to the organisation of the `Addition` phase, we will always obtain an expansion of the first type.

Proof. We proceed by induction on j. Assume that the coefficients $c_0, c_1, \ldots, c_{j-1}$ in the binary expansion of E correspond to the rule stated in the Lemma.

Suppose that iteration j is successful, i.e. condition (10) is fulfilled. Then the simple division with cutpoints x_1 and x_2 fulfils

$$U_L(0, x_1) = U_M(x_1, x_2) = \lfloor E \rfloor_{j-1} + 1/2^j, \; U_R(x_2, 1) \geq \lfloor E \rfloor_{j-1} + 1/2^j. \qquad (12)$$

Thanks to Lemma 1 the equitable value E for the players' order (L, M, R) fulfils $E \geq \lfloor E \rfloor_{j-1} + 1/2^j$ and so coefficient c_j has been correctly determined to be equal to 1.

Conversely, suppose that iteration j was unsuccessful, but still the value E of an ESD with the players' order (L, M, R) fulfils $E \geq \lfloor E \rfloor_{j-1} + 1/2^j$. Let us denote the cutpoints of one such division by x_1' and x_2'. It is clear that the leftmost point x_1 such that $U_L(0, x_1) = \lfloor E \rfloor_{j-1} + 1/2^j$ must fulfil $x_1 \leq x_1'$ and the leftmost point x_2 such that $U_M(x_1, x_2) = \lfloor E \rfloor_{j-1} + 1/2^j$ fulfils $x_2 \leq x_2'$. Since $U_R(x_2', 1) \geq \lfloor E \rfloor_{j-1} + 1/2^j$, condition (10) is also fulfilled, and we get a contradiction with the assumption that iteration j was unsuccessful. ■

The algorithm may terminate any time with an equitable division if player R says that $U_R(x_2, 1) = \lfloor E \rfloor_{j-1} + 1/2^j$. If this does not occur sufficiently early, we can proceed to `Termination` phase as soon as $1/2^j < \varepsilon$.

4.3 `Termination`

Let us summarize the situation immediately after iteration j finished and before the algorithm enters the `Termination` phase. The marks are made at points x_1 and x_2 and the values of pieces fulfil relations (12). The purpose of the `Termination` phase is to decide how to divide the cake to ensure that the values of final assignments do not differ by more than ε from each other and also from the equitable value E (notice that if the cuts were made at x_1 and x_2, player R might get too much).

A suitable division will be obtained by following the steps of the decision tree described in Figure 1. For brevity, we shall introduce the notation

$$\lfloor E \rfloor_j^+ = \lfloor E \rfloor_j + 1/2^j.$$

It is easy to see that (recall that we have the binary expansion of the first type)

$$\lfloor E \rfloor_j \leq E < \lfloor E \rfloor_j^+. \qquad (13)$$

We shall also use the symbol $\lfloor E \rfloor_j^*$ to denote an unknown (arbitrary) number in the interval $[\lfloor E \rfloor_j, \lfloor E \rfloor_j^+]$.

Theorem 3 *If* `Termination` *is entered after iteration j such that $1/2^j < \varepsilon$ then the values of the pieces assigned to players (by each one's own measure) differ by at most ε.*

1.	Ask player	R to tell the biggest point y_2 such that $U_R(y_2, 1) = \lfloor E \rfloor_j^+$.
2.	If $y_2 \leq x_2$	then cut the cake in points x_1 and x_2.
3.		(Obtained values: $\lfloor E \rfloor_j$, $\lfloor E \rfloor_j$ and $\lfloor E \rfloor_j^*$.)
4.	If $y_2 > x_2$	then ask player M to tell the biggest y_1 such that $U_M(y_1, y_2) = \lfloor E \rfloor_j^+$.
5	If $y_1 < x_1$	then cut the cake in points x_1 and y_2.
6.		(Obtained values: $\lfloor E \rfloor_j$, $\lfloor E \rfloor_j^*$ and $\lfloor E \rfloor_j^+$.)
7.		otherwise cut the cake in points y_1 and y_2
8.		(Obtained values: $\lfloor E \rfloor_j^*$, $\lfloor E \rfloor_j^+$ and $\lfloor E \rfloor_j^+$.)

Figure 1: Algorithm `Termination`

Proof. It can be easily seen that all the possibilities when trying to assign to players in the order (L, M, R) pieces with value $\lfloor E \rfloor_j^+$ have been checked. Let us review in detail what happens in which case.

Condition $y_2 \leq x_2$ means that the value of piece $[x_2, 1]$ for player R is between $\lfloor E \rfloor_j$ and $\lfloor E \rfloor_j^+$. If $y_2 > x_2$ we cut in y_2 to give player R a piece with value $\lfloor E \rfloor_j^+$ and now try to give player M a piece with value $\lfloor E \rfloor_j^+$ that finishes in y_2. This is impossible if $y_1 < x_1$, so cutting in points x_1 and y_2 gives the players the pieces with values as stated in line 6 of Figure 1. Finally, if $y_1 \geq x_1$, we know (thanks to Lemma 1) that giving to both M and R a piece with value $\lfloor E \rfloor_j^+$ (by cutting in y_1 and y_2), player L cannot obtain also so much. So making the first cutpoint in y_1 ensures for player L a value between $\lfloor E \rfloor_j$ and $\lfloor E \rfloor_j^+$.

As we entered the `Termination` phase after iteration j such that $1/2^j < \varepsilon$, we see that ε-equitability is ensured. ■

5 Generalization to any number of players

We described the Addition and the Termination phase of our ε-equitable algorithm in detail for three players. Now we explain how these ideas can be used for an arbitrary number of players.

5.1 `Addition`

In iteration j we successively issue the requests (recall that $x_0 = 0$):

For player $i = 1, 2, \ldots, n-1$:
Tell the smallest point x_i such that $U_i(x_{i-1}, x_i) = \lfloor E \rfloor_{j-1} + 1/2^j$
For player n: Is it true that $U_n(x_{n-1}, 1) \geq \lfloor E \rfloor_{j-1} + 1/2^j$?

Figure 2: Termination for n players

(If anybody thinks that $U_i(x_{i-1}, 1) < \lfloor E \rfloor_{j-1} + 1/2^j$, she says $x_i = 1$.)

For $i = n$ downto 2 do
begin
  Ask player i: tell the biggest point y_{i-1} such that $U_i(y_{i-1}, y_i) = \lfloor E \rfloor_j^+$.
  If $y_{i-1} \le x_{i-1}$ then then cut the cake in $x_1, \dots, x_{i-1}, y_i, \dots, y_{n-1}$ and STOP
end
Cut the cake in points $y_1, y_2, \dots, y_{n-1}$

Figure 3: Termination for n players

Again we say that iteration j is *successful* if the following conditions holds:

$$U_n(x_{n-1}, 1) \ge \lfloor E \rfloor_{j-1} + 1/2^j; \tag{14}$$

otherwise the iteration is said to be *unsuccessful.* Connection between the success in an iteration and the corresponding digit in the binary expansion of E is the same as that in Lemma 4; we state the assertion without proof.

Lemma 5 *For each j, the value of digit c_j in the binary expansion of the equitable value E for the players' order determined in the* `Ordering` *phase of the algorithm by the rule (11) is correct.*

5.2 Termination

`Termination` phase can be entered (unless the algorithm ends earlier with an equitable division) as soon as $1/2^j < \varepsilon$. Its purpose is to ensure that nobody will get too much. We first describe intuitively what this phase does. In the course of this phase we will have *left* players $1, 2, \dots, i-1$ who are all temporarily assigned pieces of the value exactly $\lfloor E \rfloor_j$ separated by temporary cutpoints $x_0 = 0, x_1, \dots, x_{i-1}$. *Right* players $i+1, \dots, n$ have definitive pieces with value $\lfloor E \rfloor_j^+$ and their cupoints $y_i, y_{i+1}, \dots, y_{n-1}, y_n = 1$ are definitive. Player i competes with her neighbours for piece $X = [x_{i-1}, y_i]$ whose value is not yet known, we only know that its value for player i is not smaller than $\lfloor E \rfloor_j$. At first, all players are left players, piece X touches the right-endpoint of the cake and we ask player n to tell the right-most cutpoint y_{n-1} such that the piece $[y_{n-1}, 1]$ has the value $\lfloor E \rfloor_j^+$. If $y_{n-1} \le x_{n-1}$, we know that the value of the piece $[x_{n-1}, 1]$ for player n is not bigger than $\lfloor E \rfloor_j^+$, so we can cut in points $x_1, \dots, x_{n-1}$ to achieve ε-eqitability. If $y_{n-1} > x_{n-1}$ then point y_{n-1} is made a new cupoint, player n is made a right player, the new piece X is $[x_{n-2}, y_{n-1}]$ and we ask player $n-1$ to tell the right-most cutpoint y_{n-2} such that the piece $[y_{n-2}, y_{n-1}]$ has the value $\lfloor E \rfloor_j^+$. Moving this way the piece X to the left, at latest player 1 will think that the value of piece X is smaller than $\lfloor E \rfloor_j^+$ (however, remember that not smaller than $\lfloor E \rfloor_j$) and we can cut. A formal description of this algorithm is given in Figure 3. The correctness of this procedure can easily be verified.

6 Conclusion

In this paper we proved that in spite of the positive existence results, it is impossible to find an equitable division for $n \geq 3$ players, if everybody is to receive one contiguous piece. On the other hand, we proposed a simple algorithm that finds a division such that the values assigned to players (by everybody's own measure) differ by no more than a predetermined small value.

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