P. J. ŠAFÁRIK UNIVERSITY
FACULTY OF SCIENCE
INSTITUTE OF MATHEMATICS
Jesenná 5, 041 54 Košice, Slovakia

K. Cechlárová

On coalitional resource games with shared resources

IM Preprint, series A, No. 9/2007
October 2007

On coalitional resource games with shared resources*

Katarína Cechlárová

Institute of Mathematics, Faculty of Science, P.J. Šafárik University,

Jesenná 5, 041 54 Košice, Slovakia

email: katarina.cechlarova@upjs.sk

Abstract.

Coalition formation is a popular topic studied in connection with multiagent systems. Recently, a new model of cooperation, called the Coalitional Resource Games (CRG for brief) has been introduced. In a CRG, agents wish to achieve certain goals that require expenditure of some resources. Agents form coalitions to pool their resources in order to be able to achieve a set of goals that satisfy all members of a coalition. When resources are consumable, many problems connected with CRGs are hard, e.g. Is a given coalition successful?, Is a given resource necessary for a coalition to be successful?, etc. In this paper we show a connection of CRGs with shared resources and 'max-min linear' systems of inequalities. This correspondence will enable us to derive polynomial algorithms for several problems whose counterparts for CRGs with consumable resources are hard. On the other hand, we prove that other problems are hard also in the case of shared resources.

Keywords. Multi-agent system, Cooperative game, Resources, Computational complexity.

Mathematical subjects classification. 91B68, 68Q25

1 Introduction

The study of cooperation of self-interested agents has recently attracted a lot of attention. The first game-theoretical models of cooperation of several agents studied mainly the questions connected with the ways of sharing profit obtained by or costs incurred by a common action of a group of agents [12]. However, cooperation is quite often more of a qualitative character and profit is not the

*This work was supported by the Slovak Agency for Science, contracts 1/3001/06, 1/3128/06 and Science and Technology Assistance Agency contract No. APVT-20-004104

main motivation. From the great amount of topics studied in this area let us mention here e.g. models studying cooperation of multiagent systems [17, 16, 19], location of public facilities [5], housing markets [22], formation of marriages and assigning students to colleges [11], or cooperation of patient-donor pairs in search for kidneys suitable for transplantation [14, 4].

Any sensible solution concept, capturing such notions like justice, fairness, or stability, needs to be computable, to enable the agents to actually use it [8]. Efficient computational methods are even more important, if in real situations the number of cooperating agents is very high. As examples consider the National Residence Matching Program [13] with as many as 50 000 participating students every year, kidney exchange programs (the number of patients on the waiting list for kidney transplantation in the USA is currently more than 70 000 [18]) or residence exchange fairs in Beijing with as many as 80 000 families taking part [22]. Finally, Internet, as a place for exchanging and sharing information or for the realisation of distributed computing has millions and millions of users. So no wonder that the study of computational complexity questions of problems connected with cooperative games has become very popular.

A new kind of model, called the *Qualitative Coalitional Games* was introduced by Wooldridge and Dunne [20]. The authors assumed that agents cooperate with one another in order they can mutually accomplish their goals. In a special case named the *Coalitional Resource Games* (CRG for short) [21], it is assumed that each goal requires the expenditure of a certain profile of resources and hence an incentive for an agent to join a coalition is that he may not have enough resources to achieve his goal. As an example, one may imagine collaborative science projects where a number of agents cooperate by sharing sophisticated and expensive equipment, like particle accelerators, super-computers, gene sequencers etc. Machinery can be used by several agents or for several different research projects (*shared resources*), but the character of other resources is such that as soon as one unit of them has been used for one purpose, it is no longer available for further projects. Such resources are called *consumable* and examples of them are some chemicals or biological material.

Wooldridge and Dunne [21] considered only games with consumable resources. They formulated several natural decision problems associated with them and classified their complexity. For example, SUCCESSFUL COALITION problem asks whether the pool of resources of the members of a given coalition enables them to achieve a set of goals that satisfies all of them. This problem was shown to be NP-complete. Other problem is NECESSARY RESOURCE: is it possible that a coalition will achieve its goals without the use of a given resource? This problem is co-NP-complete. MAXIMAL SUCCESSFUL COALITION problem asks whether a given coalition is successful, but after adding any other member it becomes unsuccessful. This problem is NP-complete as well as co-NP-complete. The only decision problem shown in [21] to be polynomially solvable for a CRG with consumable resources was the POTENTIAL GOAL SET: given a set of goals, does there exist a coalition such that this set of goals is both feasible for and satisfies the coalition?

In this paper we propose to formulate some of the decision problems for Coalitional Resource Games in the language of systems of linear inequalities over max-min algebra. For example, solving the SUCCESSFUL COALITION problem for a CRG with consumable resources leads to finding a solution of a system of inequalities such that some of them are linear in the usual sense and some of them are 'linear' when the operation *maximum* plays the role of addition and the operation *minimum* the role of multiplication. On the other hand, a successful coalition for a CRG with shared resources can be represented as a solution vector of a purely max-min linear system of inequalities and so its existence can be decided in polynomial time.

The organization of this paper is as follows. In Section 2 we introduce the basic concepts of the max-min algebra, review the necessary results known about solving 'one-sided systems of linear equations' and derive a new solution method for special 'two-sided systems', needed for our study. Section 3 is devoted to the Coalitional Resource Games, in particular to ones with shared resources. In Section 4 we show how to formulate several decision problems for the CRG in the language of max-min 'linear' systems and based on this correspondence we derive polynomial algorithms for them. Finally, Section 5 brings hardness proofs of some other problems.

2 Linear systems in max-min algebra

For the description of games studied in this paper we shall use systems of inequalities that are 'linear' under operations maximum and minimum replacing the classical addition and multiplication. Therefore we first review the necessary results known in this area and derive some new ones that will be used later.

Max-min algebra is a triple $\mathcal{M} = (R, \oplus, \otimes)$ where R is a linearly ordered set with the minimum and maximum elements denoted by $\mathbf{0}$ and $\mathbf{1}$, operations $\oplus =$ *maximum* and $\otimes =$ *minimum*. (By convention, the minimum of an empty set will be equal to $\mathbf{1}$.) Max-min algebra, as a special type of a semiring was introduced to model problems connected with discrete dynamic systems, synchronisation, fuzzy reasoning etc [1, 10].

In this paper, we shall take R to be equal either to the set $\mathbb{R}^+$ of nonnegative reals appended by ∞, or to the two-element Boolean algebra $\mathbb{B} = \{0, 1\}$. The symbol $\mathbb{R}^+(n, m)$ represents the set of all $n \times m$ matrices with nonnegative real entries. The set of nonnegative real vectors $\mathbb{R}^+(n, 1)$ will be denoted by $\mathbb{R}^+_n$. We use $\mathbb{B}(n, m)$ and $\mathbb{B}_n$ for binary matrices and vectors and $\mathcal{M}(n, m)$ and $\mathcal{M}_n$ for matrices and vectors over a general max-min algebra. In this paper we usually denote matrices by capitals, vectors by boldface letters and their entries by simple letters.

Given a set of vectors $S \subseteq \mathcal{M}_n$, a vector $\mathbf{x} \in S$ is said to be *maximum*, if $\mathbf{y} \leq \mathbf{x}$ holds (componentwise) for all $\mathbf{y} \in S$ and it is *maximal* if $\mathbf{y} = \mathbf{x}$ for each vector $\mathbf{y} \in S$ fulfilling $\mathbf{x} \leq \mathbf{y}$. Similarly, a vector $\mathbf{x} \in S$ is *minimal* if $\mathbf{y} = \mathbf{x}$ for each vector $\mathbf{y} \in S$ fulfilling $\mathbf{y} \leq \mathbf{x}$.

Operations $\oplus$ and $\otimes$ are extended to operations with matrices similarly as in the classical algebra. More precisely, for two compatible matrices A of type $m \times n$ and B of type $n \times p$ their (classical) product, denoted by AB, is a matrix C of type $m \times p$ such that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

On the other hand, the max-min product of matrices A and B will be denoted by $A \otimes B = C$, where

$$c_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj},$$

The basic properties of max-min linear systems of inequalities were formulated and rediscovered by many authors during the last few decades; among the first were E. Sanchez [15] and K. Zimmermann in a research report written in Czech in 1976 [23]:

Theorem 1 *Let $A \in \mathcal{M}(m,n)$, $\mathbf{b} \in \mathcal{M}_m$ be given. The maximum solution of inequality*

$$A \otimes \mathbf{x} \leq \mathbf{b} \tag{1}$$

is equal to a vector $\mathbf{x}^*(A, \mathbf{b})$ whose entries are

$$x_j^*(A, \mathbf{b}) = \min\{b_i; a_{ij} > b_i\}. \tag{2}$$

Theorem 2 *Let $A \in \mathcal{M}(m,n)$, $C \in \mathcal{M}(k,n)$, $\mathbf{b} \in \mathcal{M}_m$, $\mathbf{d} \in \mathcal{M}_k$. The system of max-min inequalities of the form*

$$A \otimes \mathbf{x} \leq \mathbf{b} \tag{3}$$
$$C \otimes \mathbf{x} \geq \mathbf{d} \tag{4}$$

is solvable if and only if vector $\mathbf{x}^*(A, \mathbf{b})$ fulfills inequality (4). Moreover, $\mathbf{x}^*(A, \mathbf{b})$ is the maximum solution of this system.

Proof. The 'if' implication is trivial. For the converse direction let us suppose that $\mathbf{y}$ is such that $A \otimes \mathbf{y} \leq \mathbf{b}$ and $C \otimes \mathbf{y} \geq \mathbf{d}$. Then $\mathbf{y} \leq \mathbf{x}^*(A, \mathbf{b})$ hence $C \otimes \mathbf{x}^*(A, \mathbf{b}) \geq \mathbf{d}$. Maximality is implied by Theorem 1. ■

Notice that since vector $\mathbf{x}^*(A, \mathbf{b})$ can be computed in $O(mn)$ time for a given matrix $A \in \mathcal{M}(m,n)$ and a given right-hand side $\mathbf{b} \in \mathcal{M}_m$, solvability of system (3)–(4) can be decided in polynomial time.

A two-sided system of the form

$$A \otimes \mathbf{x} \leq B \otimes \mathbf{y} \qquad (5)$$
$$C \otimes \mathbf{x} \geq D \otimes \mathbf{y} \qquad (6)$$

for $A \in \mathcal{M}(m,n)$, $B \in \mathcal{M}(m,p)$, $C \in \mathcal{M}(k,n)$, $D \in \mathcal{M}(k,p)$ with unknowns $\mathbf{x} \in \mathcal{M}_n$ and $\mathbf{y} \in \mathcal{M}_p$ always has a trivial, i.e. zero solution, however, a method to find nontrivial solutions has so far not been published. For two-sided systems with equations, i.e. systems of the form $A \otimes \mathbf{x} = B \otimes \mathbf{x}$ the algorithm proposed in [7] could be adapted. That algorithm was designed for a general equation of the form $F(x) = G(x)$, where F and G are residuated funtions over a partially ordered set, and thanks to F and G being 'max-min linear' here, convergence of this algorithm is ensured, but not polynomiality. Notice that for two-sided linear systems over a similar structure, called max-algebra when operation $\otimes$ is the classical addition, polynomial algorithms have recently been proposed in [6, 2].

On the other hand, we shall need max-min systems of a special form,

$$A \otimes \mathbf{x} \leq B \otimes \mathbf{y} \qquad (7)$$
$$C \otimes \mathbf{x} \geq \mathbf{y}, \qquad (8)$$

where in addition, all the matrices as well as unknown vectors $\mathbf{x}$ and $\mathbf{y}$ are required to be binary. Let us consider Algorithm 1 given in Figure 1.

Input: Matrices $A \in \mathbb{B}(p,n), B \in \mathbb{B}(p,m), C \in \mathbb{B}(m,n)$.
Output: Vectors $\mathbf{x} \in \mathbb{B}_n, \mathbf{y} \in \mathbb{B}_m$ such that $A \otimes \mathbf{x} \leq B \otimes \mathbf{y}$ and $C \otimes \mathbf{x} \geq \mathbf{y}$.
begin $k := 0;\ \mathbf{y}^k = (1,\dots,1)^T;$
  repeat $\mathbf{x}^k := \mathbf{x}^*(A, B \otimes \mathbf{y}^k);\ I^k := \{i; (C \otimes \mathbf{x}^k)_i < y_i^k\};$

    if $I^k \neq \emptyset$ **then begin** $y_i^{k+1} := \begin{cases} 0 & \text{if } i \in I^k \\ y_i^k & \text{otherwise} \end{cases};$

      $k := k+1;$
    end
  until $I^k = \emptyset$
end

Figure 1: Algorithm 1

Theorem 3 *Algorithm 1 correctly decides in $O(m(pn+pm+mn))$ time whether system* $(7)-(8)$ *has a nontrivial binary solution. Moreover, the found solution is a maximum solution of* $(7)-(8)$.

Proof. As in each **repeat-until** loop with $I^k \neq \emptyset$ at least one entry of $\mathbf{y}^k$ is switched from 1 to 0, we have at most m loops. In each loop, the computation of $\mathbf{x}^*(A, B \otimes \mathbf{y}^k)$ needs $O(pn + pm)$ steps, and $C \otimes \mathbf{x}^k$ can be computed in $O(mn)$ time. Hence the time bound follows.

To prove the correctness, denote by $\mathcal{S}$ the solution set of (7)–(8). Then notice that for each $(\mathbf{x}, \mathbf{y}) \in \mathcal{S}$, one has $(\mathbf{x}, \mathbf{y}) \leq (\mathbf{x}^0, \mathbf{y}^0)$. For the induction assumption take the following assertion:

If $I^{k-1} \neq \emptyset$, then $(\mathbf{x}, \mathbf{y}) \leq (\mathbf{x}^k, \mathbf{y}^k)$ for each $(\mathbf{x}, \mathbf{y}) \in \mathcal{S}$.

By construction, $A \otimes \mathbf{x}^k \leq B \otimes \mathbf{y}^k$. If $I^k = \emptyset$, then $(\mathbf{x}^k, \mathbf{y}^k)$ already is a solution. If not, interpreting $C \otimes \mathbf{x}^k$ as a constant right-hand side for (7) and using Theorem 1, we get that any solution $(\mathbf{x}^k, \mathbf{y}) \in \mathcal{S}$ fulfills $\mathbf{y} \leq \mathbf{y}^{k+1}$. Then, taking $B \otimes \mathbf{y}^{k+1}$ as a constant right-hand side for (8), again Theorem 1 implies that $\mathbf{x} \leq \mathbf{x}^{k+1}$ for each $(\mathbf{x}, \mathbf{y}^{k+1}) \in \mathcal{S}$. Hence, if $I^k \neq \emptyset$ then $(\mathbf{x}, \mathbf{y}) \leq (\mathbf{x}^{k+1}, \mathbf{y}^{k+1})$ for each $(\mathbf{x}, \mathbf{y}) \in \mathcal{S}$. ■

3 Coalitional Resource Games

Definition 1 *An instance of a* Coalitional Resource Game *(CRG for short) is a six-tuple* $\Gamma = (\mathcal{A}, \mathcal{G}, \mathcal{R}, D, E, T)$, *where*

- $\mathcal{A} = \{a_1, \ldots, a_m\}$ *is the set of* agents,
- $\mathcal{G} = \{g_1, \ldots, g_n\}$ *is the set of* goals,
- $\mathcal{R} = \{r_1, \ldots, r_p\}$ *is the set of* resources,
- $D \in \mathbb{B}(m, n)$ *is the* desires matrix, *with*

$$d_{ij} = \begin{cases} 1 & \text{if agent } i \text{ wishes to achieve goal } j \\ 0 & \text{otherwise,} \end{cases}$$

- $E \in \mathbb{R}^+(p, m)$ *is the* endowments matrix, *with* e_{ki} *representing the quantity of resource* k *agent* i *is endowed with,*
- $T \in \mathbb{R}^+(p, n)$ *is the* technology matrix, *with* t_{kj} *representing the quantity of resource* k *needed to achieve goal* j.

Definition 2 *For a nonempty coalition* $Y \subseteq \mathcal{A}$ *we say that a set of goals* X *is* satisfying, *if for each* $a_i \in Y$ *there exists a goal* $g_j \in X$ *such that* $d_{ij} = 1$. *The family of all satisfying sets of goals for a given coalition* Y *will be denoted by* $\mathcal{D}(Y)$. *A set of goals* X *is* feasible *for a given nonempty coalition* $Y \subseteq \mathcal{A}$, *if coalition* Y *has enough resources to achieve each goal from* X. *The family of all feasible sets of goals for a given coalition* Y *will be denoted by* $\mathcal{F}(Y)$. *A coalition* Y *is said to be* successful, *if* $\mathcal{D}(Y) \cap \mathcal{F}(Y) \neq \emptyset$.

Similarly as in [21], we suppose that each agent i wishes to achieve any goal from the set $\mathcal{G}_i = \{g_j \in \mathcal{G}; d_{ij} = 1\}$, he is indifferent between them and obtains no extra utility from achieving more than one goal. However, one can be interested in inclusion-maximal or inclusion-minimal achievable sets of goals.

Definition 3 *Resource r_k is called* consumable*, if any unit of it used for one goal, cannot be used for another one. A CRG is called a* CRG with consumable resources*, if each resource $r_k \in \mathcal{R}$ is consumable. Resource r_k is called* shared*, if it can be used for all goals simultaneously without restriction. A CRG is called a* CRG with shared resources*, if each resource $r_k \in \mathcal{R}$ is shared. A CRG is called* binary*, if all the entries of matrices E and T are either 0 or 1.*

In what follows, we shall assume that all the considered CRGs with shared resources are binary, i.e. each goal either requires a particular resource or not and similarly, a particular agent either owns the resource or not. Further, we shall also suppose that a feasible goal is available to all members of a particular successful coalition, i.e. that all goals are 'shared' in the sense that their availability by the members of a coalition is not dependent on the number of agents in it. On the other hand, we do not place similar restrictions on CRGs with consumable resources.

Example 1 *Let us consider a CRG with three agents, two goals and four resources, i.e. $m = 3$, $n = 2$ and $p = 4$. The desires, endowment and technology matrices are as follows:*

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let us first suppose that all the resources are consumable. For this CRG no successful coalition exists, as we now show.

- *If a coalition does not contain agent a_3, it is deemed to be unsuccessful, as each goal requires resource r_4 with which only agent a_3 is endowed.*
- *Any coalition containing agent a_3 must achieve goal g_2 to satisfy him. However, this goal requires 2 units of resource r_2, which only agent a_1 has and 1 unit of resource r_3, that is owned only by agent a_2. Hence, the only possibility is the grandcoalition. However, coalition $Y = \{a_1, a_2, a_3\}$ must achieve both goals, for which 2 units of resourse r_4 are needed. As all agents together have only 1 unit of this resourse, the grandcoalition is not successful.*

In all the following examples we shall suppose that the resources are shared.

Example 2 *Now let again $m = 3$, $n = 2$ and $p = 4$. We take binary matrices*

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Similarly as before, any successful coalition must contain agent a_3 (because of resource r_4) and then also agents a_1 and a_2 to ensure the availability of resources r_2 and r_3 for goal g_2. But now the grandcoalition is successful (the only successful coalition in this game), as it has all the necessary resources to achieve both goals. Further, it is easy to see that the only satisfying and feasible goal set for the grandcoalition is $\mathcal{G}$ itself and that this coalition needs all the resources for its success.

Example 3 *The nonconsumable nature of resources as well as goals implies that the union of two successful coalitions is also successful However, the intersection of two successful coalitions need not be successful. Take $\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{G} = \{g_1\}$, $\mathcal{R} = \{r_1, r_2\}$,*

$$D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Both resources are necessary to to achieve the only goal in this game. Coalitions $\{a_1, a_2\}$ as well as $\{a_1, a_3\}$ are successful, but their intersection, the singleton coalition $\{a_1\}$ is not.

Example 4 *Being successful is not a monotone property. This means that there may exist two successful coalitions $Y_1 \subset Y_2$ as well as an unsuccessful coalition Y_3 with $Y_1 \subset Y_3 \subset Y_2$. Take $\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{G} = \{g_1, g_2, g_3\}$, $\mathcal{R} = \{r_1, r_2, r_3\}$ and*

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Coalitions $\{a_1\}$ and $\{a_1, a_2, a_3\}$ are both successful – the former one is happy with goal g_1 that requires only resource r_1 and agent a_1 owns this resource; the latter one has enough resources to achieve all goals. However, neither from coalitions $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$ is successful. The first of them needs goal g_2 but does not possess resource r_3, in the second one agent a_3 wishes goal g_3 but this coalition does not own resource r_2 that is needed, and finally the third coalition do not have at their disposal resource r_1, needed for all goals.

If one uses just definitions, then checking whether a given coalition is successfull may in general require to consider all the possible subsets of $\mathcal{G}$; checking whether a given successful coalition does not admit a proper subcoalition that is also successful may mean (see the previous example) the necessity of checking all its subcoalitions, so the hardness results of [21] are not a very great surprize. However, the hardness results do not carry over to the case with shared resources. In the following section we shall obtain polynomial algorithms for several problems formulated for such CRGs using their relation with max-min linear systems.

Instead of coalition $Y \subseteq \mathcal{A}$ we shall often take its characteristic vector $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{B}_m$ defined by

$$y_i = \begin{cases} 1 & \text{if } a_i \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, a set of goals $X \subseteq \mathcal{G}$ is represented by its characteristic vector $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{B}_n$ with

$$x_j = \begin{cases} 1 & \text{if } g_j \in X \\ 0 & \text{otherwise.} \end{cases}$$

4 CRGs and max-min linear systems

Our first considered decision problem is the following:

SUCCESSFUL COALITION
Instance: CRG Γ and a nonempty coalition $Y \subseteq \mathcal{A}$.
Question: Is Y successful?

We show the connection between SUCCESSFUL COALITION and systems of linear inequalities.

Theorem 4 *Let Γ be a CRG with consumable resources and $Y \subseteq \mathcal{A}$ a nonempty coalition. Coalition Y is successful if and only if there exists a vector $\mathbf{x} \in \mathbb{B}_n$ such that*

$$D \otimes \mathbf{x} \geq \mathbf{y} \tag{9}$$

$$T\mathbf{x} \leq E\mathbf{y}. \tag{10}$$

Proof. Realize that the i^{th} inequality in (9) says

$$\max_{j=1,2,\dots,n} min\{d_{ij}, x_j\} \geq y_i,$$

which can be reworded in the following way: if agent a_i belongs to coalition Y, then there exists a goal $g_j \in X$, which this agent desires. On the other hand, the k^{th} inequality in (10) says

$$\sum_{j=1}^{n} t_{kj} x_j \leq \sum_{i=1}^{m} e_{ki} y_i,$$

hence its left-hand side counts the amount of resource r_k needed to achieve all the goals in X and the right-hand side expresses the amount of resource r that coalition Y owns in common. Hence altogether, inequality (10) ensures that coalition Y has enough resources to bring about the set of goals X. ■

Wooldridge and Dunne [21] showed that SUCCESSFUL COALITION is an NP-complete problem for binary CRG with consumable resources. (Notice that a polynomial algorithm for a system of the form (9–10) is not known.) For CRGs with shared resources we show the connection of successful coalitions and 'pure' max-min linear systems of inequalities. As such systems are polynomially solvable, we shall also be able to derive computational results for some other problems connected with coalitions. Their formulations will be given later.

Theorem 5 *Let Γ be a CRG with shared resources and $Y \subseteq \mathcal{A}$ a nonempty coalition. Coalition Y is successful if and only if there exists a vector $\mathbf{x} \in \mathbb{B}_m$ such that for the characteristic vectors $\mathbf{x}$ and $\mathbf{y}$ the following holds:*

$$D \otimes \mathbf{x} \geq \mathbf{y} \tag{11}$$
$$T \otimes \mathbf{x} \leq E \otimes \mathbf{y} \tag{12}$$

Proof. The meaning of inequality (11) is the same as in Theorem 4. The k^{th} inequality in (12) now says

$$\max_{j=1,2,\dots,n} min\{t_{kj}, x_j\} \leq \max_{i=1,2,\dots,m} min\{e_{ki}, y_i\},$$

hence the left-hand side is equal to 1 if and only if some goal in set X requires resource r_k and its right-hand side is equal to 1 if and only if at least one member of coalition Y owns this resource. Summing up, inequality (12) ensures that coalition Y has all resources needed to achieve the set of goals X. ■

As for a given coalition Y the right-hand sides in (11)–(12) are constant vectors, using Theorem 2 we have

Corollary 1 SUCCESSFUL COALITION *problem for CRG with shared resources can be decided in polynomial time, moreover, the inclusion-maximum set of goals in $\mathcal{F}(Y) \cap \mathcal{D}(Y)$ can be found.*

Now we turn to the following problem:

> SUCCESSFUL COALITION EXISTENCE
> *Instance:* CRG Γ with shared resources.
> *Question:* Does there exist a successful coalition $Y \subseteq \mathcal{A}$?

Theorem 5 still applies, but the righ-hand sides of inequalities (11)–(12) are no longer constants. So we have a two-sided system of max-min linear inequalities of the form considered in Section 2 and we can use Algorithm 1.

Theorem 6 *An instance of* SUCCESSFUL COALITION EXISTENCE *is a 'yes' instance if and only if the corresponding system* $(11)-(12)$ *with unknowns* $\mathbf{x}$ *and* $\mathbf{y}$ *has a nontrivial solution. Hence, this problem can be decided in polynomial time.*

So far we know that if a CRG with shared resources has a successful coalition then there is one inclusion-maximum successful coalition, but it is an easy exercise to construct examples with several inclusion-minimal successful coalitions. One can formulate the following problem:

> MINIMAL SUCCESSFUL COALITION
> *Instance:* CRG Γ with shared resources, successful coalition $Y \subseteq \mathcal{A}$.
> *Question:* Is Y minimal, i.e. is it true that each nonempty proper subcoalition of Y is unsuccessful?

Theorem 7 MINIMAL SUCCESSFUL COALITION *can be decided in polynomial time.*

Proof. For a given successful coalition Y and its characteristic vector $\mathbf{y}$, system $(11)-(12)$ with the unknown vector $\mathbf{x}$ is solvable. Let us consider for each i such that $a_i \in Y$ system $(11)-(12)$ with one extra inequality of the form

$$y_i \leq 0.$$

(For brevity, this system will be called an i-system). Clearly, an i-system is a one-sided max-min linear system, so can be solved in polynomial time. Moreover, it is solvable if and only if there exists a successful subcoalition of Y not containing agent a_i. Hence Y is a minimal successful coalition if and only if none of the $|Y|$ i-systems is solvable, which can be decided in polynomial time. ■

A converse scenario can also be thought of. Suppose that a central body (say a goverment) is interested in a number of research projects, which could be followed by various research groups (universities, faculties or research institutes). However, the research groups are autonomous in the sense that they will choose independently from the central body which projects to follow and whether to cooperate with other research groups. The question is whether there could exist a consorcium of the existing research groups able and willing to accomplish all the intended research projects.

> POTENTIAL GOAL SET
> *Instance:* CRG Γ, a set of goals $X \subseteq \mathcal{G}$.
> *Question:* Is there a coalition $Y \subseteq \mathcal{A}$ such that $X \in \mathcal{F}(Y) \cap \mathcal{D}(Y)$?

Theorem 8 POTENTIAL GOAL SET *problem is polynomially solvable for each CRG with shared resources.*

Proof. A set X is a potential goal set if and only if the system of max-min linear inequalities (11)–(12) with vector $\mathbf{x}$ being the characteristic vector of the set X and unknown vector $\mathbf{y}$ is solvable. As this is a one-sided max-min linear system, POTENTIAL GOAL SET is polynomially solvable. ■

The following problems deal with resources.

NECESSARY RESOURCE FOR COALITION
Instance: CRG Γ, successful coalition $Y \subseteq \mathcal{A}$, resource $r_k \in \mathcal{R}$.
Question: Is resource r_k necessary for Y, i.e. is it true that $(T \otimes \mathbf{x})_r > 0$ for each set of goals $X \in \mathcal{D}(Y) \cap \mathcal{F}(Y)$?

NECESSARY RESOURCE
Instance: CRG Γ and resource $r_k \in \mathcal{R}$.
Question: Is resource r_k necessary for success, i.e. is it true that $(T \otimes \mathbf{x})_r > 0$ for each successful coalition $Y \subseteq \mathcal{A}$ and for each set of goals $X \in \mathcal{D}(Y) \cap \mathcal{F}(Y)$?

Theorem 9 *Let Γ be a CRG with shared resources, $Y \subseteq \mathcal{A}$ a nonempty coalition and r_k a resource. Coalition Y is successful and resource r_k is necessary for Y if and only if system* (11) − (12) *is solvable, but the following system is not:*

$$D \otimes \mathbf{x} \geq \mathbf{y} \tag{13}$$
$$T \otimes \mathbf{x} \leq E \otimes \mathbf{y} \tag{14}$$
$$(T \otimes \mathbf{x})_k \leq 0 \tag{15}$$

Proof. Inequalities (13)–(14) ensure that Y is a successful coalition. Hence when (13)–(15) is not solvable, then it is exactly because of for no set of goals $X \in \mathcal{D}(Y) \cap \mathcal{F}(Y)$ the use of resource r_k can be avoided. ■

For both problems formulated for resources, Theorem 9 gives a connection with max-min linear systems. As system (13)–(15) is either one-sided (for NECESSARY RESOURCE FOR COALITION) or of the special form solvable by Algorithm 1, we have

Corollary 2 *Problems* NECESSARY RESOURCE *as well as* NECESSARY RESOURCE FOR COALITION *for CRG with shared resources can be decided in polynomial time. Moreover, if the given resource is not necessary, the inclusion-maximal set of goals achievable without this resource can be found in polynomial time.*

5 Hard problems

In spite of the polynomiality of the problems considered in the previous section, a complete description of the structure of all successful coalitions will not be so easy. Here we prove some hardness results for CRGs with shared resources.

SUCCESSFUL COALITION SPLITTING
Instance: CRG Γ with shared resources, successful coalition $Y \subseteq \mathcal{A}$.
Question: Is it possible to split Y into two successful subcoalitions?

Theorem 10 SUCCESSFUL COALITION SPLITTING *problem is NP-complete.*

Proof. This problem is in the class NP, due to Corollary 1. To prove completeness, we shall construct a polynomial transformation from the following NP-complete problem [9, Problem SP4]:

SET SPLITTING
Instance: A collection $\mathcal{C}$ of subsets of a finite set S.
Question: Is it possible to split S into two subsets S_1, S_2 in such a way that $S_1 \cap C \neq \emptyset$ as well as $S_2 \cap C \neq \emptyset$ for each $C \in \mathcal{C}$?

So let $(S, \mathcal{C})$ be an instance of SET SPLITTING. In the corresponding CRG Γ there will be one agent a_i for each element $s_i \in S$ and one resource r_j for each $C_i \in \mathcal{C}$. Agent a_i owns precisely those resources r_j for which $s_i \in C_j$. There is just one goal g and it requires all resources.

Clearly, the grandcoalition $Y = \mathcal{A}$ is successful in Γ. Moreover, Y can be split into two successful coalitions if and only if $(S, \mathcal{C})$ is a yes-instance of SET SPLITTING. ■

Our last two problems deal with cardinality of successful coalition and satisfying goal set, respectively.

MINIMUM CARDINALITY SUCCESSFUL COALITION
Instance: CRG Γ with shared resources, an integer k.
Question: Does Γ admit a successful coalition containing at most k agents?

Theorem 11 MINIMUM CARDINALITY SUCCESSFUL COALITION *problem is NP-complete.*

Proof. Thanks to Corollary 1, this problem is in the class NP. Now we construct a polynomial transformation from Problem SP8 of [9]:

HITTING SET
Instance: A collection $\mathcal{C}$ of subsets of a finite set S, an integer k.
Question: Does there exist a subset $S' \subseteq S$ such that $|S'| \leq k$ and $S \cap C \neq \emptyset$ for each $C \in \mathcal{C}$?

Let $(S, \mathcal{C}, k)$ be an instance of HITTING SET, let us define an instance (Γ, k) of MINIMUM CARDINALITY SUCCESSFUL COALITION as follows: there will be one agent a_i for each element $s_i \in S$, one resource r_j for each set $C_j \in \mathcal{C}$ and one goal g, requiring all resources. Moreover, we suppose that each agent a_i owns precisely those resources r_j for which $s_i \in C_j$. It is trivial to see that a coalition Y is successful if and only if the corresponding subset $S' = \{s_i \in S; a_i \in Y\}$ is a hitting set so we are ready. ■ HITTING SET will be used in the NP-completeness proof of the following problem too:

> MINIMUM CARDINALITY GOAL SET
> *Instance:* CRG Γ with shared resources, successful coalition $Y \subseteq \mathcal{A}$, integer k.
> *Question:* Is there a goal set $X \in \mathcal{D}(Y) \cap \mathcal{F}(Y)$ with $|X| \leq k$?

Theorem 12 MINIMUM CARDINALITY GOAL SET *problem is NP-complete.*

Proof. It is easy to see that this problem is in the class NP. To prove completeness, take an instance $(S, \mathcal{C}, k)$ of HITTING SET and define an instance (Γ, Y, k) of MINIMUM CARDINALITY GOAL SET in the following way: there is one agent a_i for each set $C_i \in \mathcal{C}$, one goal g_j for each element $s_j \in S$. Agent a_i wishes to achieve exactly the goals g_j such that $s_j \in C_i$. Further, there is only one resource r, each goal requires it and each agent possesses it. Now it is easy to see that the grandcoalition $Y = \mathcal{A} = \{a_i; C_i \in \mathcal{C}\}$ is successful and the goal sets in $\mathcal{D}(Y) \cap \mathcal{F}(Y)$ are precisely the hitting sets of $(S, \mathcal{C}, k)$. ■

6 Conclusion

The aim of this paper was to complement the theory presented in [21] by concentrating on Coalitional Resource Games with shared resources. As far as we know, such games have not been considered in literature yet. We revealed the relation of several decision problems for these game and systems of max-min linear inequalities, a well established and quite developed theory nowadays. This correspondence enabled us to derive easily many polynomial algorithms for problems whose counterparts are hard for CRGs with consumable resources. On the other hand, we also showed that there are computationally difficult problems in this area too.

References

[1] F. L. Baccelli, G. Cohen, G.J. Olsder, J.P. Quadrat, *Synchronization and Linearity, An Algebra for Discrete Event Systems*, Wiley, Chichester, 1992.

[2] P. Butkovič, K. Zimmermann, *A strongly polynomial algorithm for solving two-sided linear systems in max-algebra*, Discrete Appl. Math. **154** (2006), 437-446.

[3] K. Cechlárová, *Solution of interval linear systems in max-plus algebra*, Proc. SOR 2001, Eds. V. Rupnik, L. Zadnik-Stirn, S. Drobne, Preddvor, Slovenia, 321–326.

[4] K. Cechlárová, T. Fleiner and D. Manlove, *The kidney exchange game*, Proc. SOR '05 (2005), Eds. L. Zadnik-Stirn, S. Drobne, 77-83.

[5] L. Chen, X. Deng, Q. Fang and F. Tian, *Condorcet Winners for Public Goods*, Annals of Operations Research **137** (2005), 229-242.

[6] R.A. Cuninghame-Green and P. Butkovič, *The equation* $A \otimes x = B \otimes y$ *over* $(\{-\infty\} \cup \mathbf{R}, max, +)\}$, Theoretical Computer Science **293**(2003), 3–12.

[7] R.A. Cuninghame-Green, K. Zimmermann, *Equation with residuated function*, Comment. Math. Univ. Carolinae **42** (2001), 729–740.

[8] X. Deng and C.H. Papadimitriou, *On the complexity of cooperative solution concepts*, Mathematics of Operations Research **19**(1994) 257–266.

[9] M.R. Garey and D.S. Johnson, *Computers and Intractability*, Freeman, San Francisco (1979).

[10] J.S. Golan, *Semirings and their applications*, Kluwer, Dordrecht (1999).

[11] D. Gusfield and R. W. Irving, *The Stable Marriage Problem: Structure and Algorithms*, Foundations od Computing, MIT Press, Cambridge (1989).

[12] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press (1944).

[13] National Residents Matchnig Program, http://www.nrmp.org.

[14] A. Roth, T. Sönmez and U. Ünver, *Pairwise Kidney Exchange*, J. Econ. Theory, Journal of Economic Theory **125/2**, 151–188 (2005).

[15] E. Sanchez, *Resolution of composite fuzzy relation equations,* Inform. and Control **30**(1976), 38–48.

[16] T. Sandholm, K. Larson, M. Andersson, O. Shehory and F. Tohm, *Coalition structure generation with worst case guarantees*, Artificial Intelligence **111**(1–2) (1999), 209–238.

[17] O. Shehory and S. Kraus, *Methods for task allocation via agent coalition formation*, Artificial Intelligence **101** (1–2) (1998), 165-200.

[18] United Network for Organ Sharing, http://www.unos.org.

[19] M. Wooldridge, *An Introduction to Multiagent Systems*, John Wiley & Sons, New York (2002).

[20] M. Wooldridge and P.E. Dunne, *On the computational complexity of qualitative coalitional games*, Artificial Intelligence **158**(2004) 27–73.

[21] M. Wooldridge and P.E. Dunne, *On the computational complexity of coalitional resource games*, Artificial Intelligence **170**(2006) 835–871.

[22] Y. Yuan, *Residence exchange wanted: A stable residence exchange problem*, European Journal of Operational Research **90**(1996) 536–546.

[23] K. Zimmermann, *Extremální algebra*, Ekon. ústav ČSAV, Praha (1976) (in Czech).

Recent IM Preprints, series A

2003

1/2003 Cechlárová K.: *Eigenvectors of interval matrices over max-plus algebra*
2/2003 Mihók P. and Semanišin G.: *On invariants of hereditary graph properties*
3/2003 Cechlárová K.: *A problem on optimal transportation*

2004

1/2004 Jendroľ S. and Voss H.-J.: *Light subgraphs of graphs embedded in the plane and in the projective plane* – survey
2/2004 Drajnová S., Ivančo J. and Semaničová A.: *Numbers of edges in supermagic graphs*
3/2004 Skřivánková V. and Kočan M.: *From binomial to Black-Scholes model using the Liapunov version of central limit theorem*
4/2004 Jakubíková-Studenovská D.: *Retracts of monounary algebras corresponding to groupoids*
5/2004 Hajduková J.: *On coalition formation games*
6/2004 Fabrici I., Jendroľ S. and Semanišin G., ed.: *Czech – Slovak Conference GRAPHS 2004*
7/2004 Berežný Š. and Lacko V.: *The color-balanced spanning tree problem*
8/2004 Horňák M. and Kocková Z.: *On complete tripartite graphs arbitrarily decomposable into closed trails*
9/2004 van Aardt S. and Semanišin G.: *Non-intersecting detours in strong oriented graphs*
10/2004 Ohriska J. and Žulová A.: *Oscillation criteria for second order non-linear differential equation*
11/2004 Kardoš F. and Jendroľ S.: *On octahedral fulleroids*

2005

1/2005 Cechlárová K. and Vaľová V.: *The stable multiple activities problem*
2/2005 Lihová J.: *On convexities of lattices*
3/2005 Horňák M. and Woźniak M.: *General neighbour-distinguishing index of a graph*
4/2005 Mojsej I. and Ohriska J.: *On solutions of third order nonlinear differential equations*
5/2005 Cechlárová K., Fleiner T. and Manlove D.: *The kidney exchange game*
6/2005 Fabrici I., Jendroľ S. and Madaras T., ed.: *Workshop Graph Embeddings and Maps on Surfaces 2005*
7/2005 Fabrici I., Horňák M. and Jendroľ S., ed.: *Workshop Cycles and Colourings 2005*

2006

1/2006 Semanišinová I. and Trenkler M.: *Discovering the magic of magic squares*
2/2006 Jendroľ S.: *NOTE – Rainbowness of cubic polyhedral graphs*
3/2006 Horňák M. and Woźniak M.: *On arbitrarily vertex decomposable trees*
4/2006 Cechlárová K. and Lacko V.: *The kidney exchange problem: How hard is it to find a donor ?*

5/2006 Horňák M. and Kocková Z.: *On planar graphs arbitrarily decomposable into closed trails*

6/2006 Biró P. and Cechlárová K.: *Inapproximability of the kidney exchange problem*

7/2006 Rudašová J. and Soták R.: *Vertex-distinguishing proper edge colourings of some regular graphs*

8/2006 Fabrici I., Horňák M. and Jendroľ S., ed.: *Workshop Cycles and Colourings 2006*

9/2006 Borbeľová V. and Cechlárová K.: *Pareto optimality in the kidney exchange game*

10/2006 Harminc V. and Molnár P.: *Some experiences with the diversity in word problems*

11/2006 Horňák M. and Zlámalová J.: *Another step towards proving a conjecture by Plummer and Toft*

12/2006 Hančová M.: *Natural estimation of variances in a general finite discrete spectrum linear regression model*

2007

1/2007 Haluška J. and Hutník O.: *On product measures in complete bornological locally convex spaces*

2/2007 Cichacz S. and Horňák M.: *Decomposition of bipartite graphs into closed trails*

3/2007 Hajduková J.: *Condorcet winner configurations in the facility location problem*

4/2007 Kovárová I. and Mihalčová J.: *Vplyv riešenia jednej difúznej úlohy a následný rozbor na riešenie druhej difúznej úlohy o 12-tich kockách*

5/2007 Kovárová I. and Mihalčová J.: *Prieskum tvorivosti v žiackych riešeniach vágne formulovanej úlohy*

6/2007 Haluška J. and Hutník O.: *On Dobrakov net submeasures*

7/2007 Jendroľ S., Miškuf J., Soták R. and Škrabuláková E.: *Rainbow faces in edge colored plane graphs*

8/2007 Fabrici I., Horňák M. and Jendroľ S., ed.: *Workshop Cycles and Colourings 2007*

Preprints can be found in: http://umv.science.upjs.sk/preprints